SOME INVARIANT SOLUTIONS OF KORTEWEG-DE VRIES EQUATIONS

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The Lie groups obtained here are used for finding the invariant solutions of the equations. One class of solutions, expressible in terms of Painlevé transcendental functions, is connected with the initial development stage of a disturbance in a dispersive medium.

To discover the role of dispersion in nonlinear processes and the nature of the relevant phenomena, the Korteweg-de Vries equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \beta\frac{\partial^3 u}{\partial x^3} = 0 \tag{0.1}$$

has come to be widely used in recent years [1–6].

It is obtained when the dispersion law is almost linear and is expressible as

$$\omega = v_f k\,(1 + \delta k^2) \tag{0.2}$$

where v_f is the phase velocity of the small oscillations $\delta k^2 \ll 1$, and δ is a constant characterizing the size of the dispersion effects. In this case, the fundamental equations of a wide range of problems with finite but small disturbances (e. g., the case of surface waves in shallow water [1], or plasma waves [2, 6]) reduce to (0.1). The solutions of this equation have been investigated both numerically [2–5] and by means of the approximate analytic method developed by Whitham [7], in [3,4].

In this paper we investigate the group properties of Eq. (0.1) and find the infinitesimal operators generating the Lie algebra of the fundamental group, with the result that invariant [9] solutions can be obtained. In addition to the familiar solutions (solitons, periodic waves [1,3–5], and similarity solutions [2–4]), new previously discussed solutions of (0.1) are thus obtained.

In addition, two further equations of the type (0.1) with corresponding dispersion laws will be considered, namely,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \beta\frac{\partial^3 u}{\partial t\partial x^2} = 0, \qquad \omega = \frac{v_f k}{1+\delta k^2} \tag{0.3}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \beta u\frac{\partial^3 u}{\partial t^3} = 0, \qquad \omega = v_f k + \delta\omega^3 \tag{0.4}$$

The invariant solutions of these equations are analyzed in sections 2 and 3 of this paper.

1. The concept of invariant solutions is described in [8,9], together with a method whereby sets of particular solutions may be found. It can be shown by using this method that the Lie algebra of the fundamental group of Eq. (0.1) is generated by the infinitesimal operators

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}$$
$$X_4 = x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u} \tag{1.1}$$

These operators correspond to the following transformations: X_1 to a time translation, X_2 to a coordinate translation, X_3 to a Galilean displacement, and X_4 to a dilatation with respect to all the variables.

To find the essentially distinct invariant solutions, we need to know all the dissimilar subgroups of the fundamental group. All the remaining invariant solutions are obtained by transformation of the functions and variables, and no integration of differential equations is required [8, 9]. The optimum system of one-parameter subgroups is

$$X_1,\ X_2,\ X_3,\ X_4,\ X_5 = X_1 + \alpha X_3 \tag{1.2}$$

where α is an arbitrary constant.

For the subgroup X_1 we have two invariants $I_1 = x$, $I_2 = u$. The solution is sought in the form $u = u(x)$ and is expressible in terms of Jacobi elliptic functions [10]. The transformation $x_1 = x - vt$ gives a class of solutions corresponding to nonlinear periodic waves, which have already been widely investigated:

$$u = v + \frac{2}{3}a\left(1 + \frac{1}{s^2}\right) - 2a\,sn^2\left(\frac{\sqrt{a}}{\sqrt{6\beta}}\frac{x - vt}{s},\ s\right) \tag{1.3}$$

Here, a is the wave amplitude and s the modulus of the elliptic function. Notice that here and below, subgroups leading to nontrivial solutions are considered.

The dilatation group gives the similarity solutions [9]. The invariants of X_4 are

$$I_1 = x^3 / t, \qquad I_2 = ut^{2/3}$$

The solution is sought as

$$u = (\beta(3t)^{-2})^{1/3}U(\lambda), \qquad \lambda = (3\beta)^{1/3}xt^{-1/3}$$

and satisfies the equation

$$\frac{d^3U}{d\lambda^3} + U\frac{dU}{d\lambda} - \lambda\frac{dU}{d\lambda} - 2U = 0 \tag{1.4}$$

The importance of the similarity solutions in the study of the asymptotic development of a disturbance in a dispersive medium was revealed in great detail in [4].

The phenomenon of moving singularities is typical of nonlinear equations [11]. For instance, (1.4) has a moving second-order pole

$$U = -\frac{12}{(\lambda - \lambda_0)^2} + \lambda_0 + a(\lambda - \lambda_0)^2 - \frac{\lambda_0}{3}(\lambda - \lambda_0)^3 + h(\lambda - \lambda_0)^4$$
$$- \frac{6a + 5\lambda_0}{216}(\lambda - \lambda_0)^6 + \frac{5a\lambda_0 + 18h}{450}(\lambda - \lambda_0)^7 - \frac{\lambda_0^2 + 18ah}{792}(\lambda - \lambda_0)^8 + \ldots \tag{1.5}$$

Here, λ_0, a, and h are arbitrary constants of integration.

Noticé that, when account is taken of dispersion, i.e., in (0.1), we have to introduce a further term $\nu d^2u/dx^2$; then the equation will not admit the dilatation group X_4.

The most interesting solutions of (0.1) are those which are determined by the dispersion (they must contain the dispersion parameter β) and retain their functional dependence on x and t when allowance is made for dissipation. Such solutions include, in addition to those given by the subgroup X_1, the invariant solutions given by the subgroup X_5. The invariants of X_5 are $I_1 = x - \frac{1}{2}\alpha t^2$, $I_2 = u - \alpha t$. We seek the solution in the form $u = \alpha t + U(\lambda)$, where $\lambda = x - \frac{1}{2}\alpha t^2$:

$$\beta\frac{d^3U}{d\lambda^3} + U\frac{dU}{d\lambda} + \alpha = 0 \tag{1.6}$$

Equation (1.6) admits the transformation $x_1 = x + a$, so that the constant of integration in the first integration of (1.6) can be taken to be zero. On putting

$$U(\lambda) = -(12^3\alpha^2\beta)^{1/5}W, \qquad \lambda = (12\alpha^{-1}\beta^2)^{1/5}z$$

we get

$$\frac{d^2W}{dz^2} = 6W^2 + z \tag{1.7}$$

The essentially transcendental Painlevé function [11] is a solution of this equation. While the solution contains no moving critical points, it has a moving pole:

$$W = \frac{1}{(z-z_0)^2} - \frac{z_0}{10}(z-z_0)^2 - \frac{1}{6}(z-z_0)^3 + h(z-z_0)^4 + \frac{z_0^2}{300}(z-z_0)^6 + \ldots$$

The physical significance of this solution may be seen by considering the limiting case $\beta \to 0$. Here $u = \alpha t - \sqrt{\alpha^2 t^2 - 2\beta x}$ gives the development of the shock wave from a parabolic profile, and the constant α is the radius of curvature of the nose of the parabola.

The dispersion of the medium results in a modulation wave moving along a smooth profile and reaching the point $x - {}^1/_2\alpha t^2$, where α is the minimum radius of curvature of the profile. This is in accordance with the computations of [4], which show that the asymptotic development of the initial disturbance leads, on the basis of (0.1), to the formation of a series of solitons with an oscillating tail.

2. Equation (0.1) is used for problems with initial data, since the solution u(x, t) is uniquely determined by the initial disturbance. For Eq. (0.3), the initial disturbance again uniquely determines the solution, provided we assume, as is entirely justified physically, that the propagation speed of the disturbance is finite. But the inclusion of the term $\nu\partial^2 u/\partial x\partial t$ in (0.3) does not lead to a contraction of the Lie group, and hence does not change the number of its invariant solutions.

Equation (0.3) has a three-parameter Lie group. The Lie algebra is generated by the infinitesimal operators

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} \tag{2.1}$$

These operators correspond to the following transformations: X_1 is a time translation, X_2 is a coordinate translation, and X_3 is a dilatation. Working similar to that of the previous section shows that the optimum system of one-parameter subgroups can be written as

$$X_1,\ X_2,\ X_3,\ X_4 = \alpha X_2 + X_3,\ X_5 = X_1 + \alpha X_2 \tag{2.2}$$

The subgroups X_1 and X_2 yield trivial solutions.

The similarity solution is given by the dilatation group. The invariants of the operator X_3 are $I_1 = x$, $I_2 = ut$. The solution is sought as $u = U(x)/t$. It satisfies the equation

$$\beta\frac{d^2U}{dx^2} - U\frac{dU}{dx} + U = 0 \tag{2.3}$$

The parametric solution of (2.3) reduces to a single quadrature:

$$x = \frac{1}{2}\sqrt{2\beta}\int\frac{(p+1)\,dp}{p^2(p+\ln p+c)^{1/2}}$$
$$U = \sqrt{2\beta}\,(p+\ln p+c)^{1/2} \tag{2.4}$$

where p is a parameter; the second constant of integration in (2.3) is given by the admissible transformation $x_1 = x + a$. The subgroup X_4 gives solutiosn which in the limit become the similarity solution. The invariants of X_4 are $I_1 = x - \alpha \ln t$, $I_2 = ut$. The solution is sought in the form

$$u = U(\lambda)/t, \quad \lambda = x - \alpha\ln t$$

As a result, we obtain

$$\beta\left(\alpha\frac{d^3U}{d\lambda^3} + \frac{d^2U}{d\lambda^2}\right) - U\frac{dU}{d\lambda} + U - \alpha\frac{dU}{d\lambda} = 0 \tag{2.5}$$

The order of the equation can be lowered by the standard substitution $dU/d\lambda = p(U)$. While the resulting equation does not admit a first integral, it can be seen by writing its general solution as a series that this solution is not an algebraic function of the two constants of integration. The general solution is an essentially transcendental function of the two constants, different from the transcendental functions defined by first-order equations with algebraic coefficients.

The subgroup X_5 gives a nonlinear traveling wave. The invariants of X_5 are $I_1 = u$, $I_2 = x - \alpha t$. The solution is sought in the form $u(\lambda)$. As in the case of (1.3), it is expressible in terms of a Jacobi elliptic function:

$$u = \alpha - {}^2/_3\, a\,(1 + s^{-2}) + 2a\,\mathrm{sn}^2\left(\frac{\sqrt{a}}{\sqrt{6\beta\alpha}}\,\frac{x - \alpha t}{s},\ s\right) \tag{2.6}$$

The difference between (2.6) and (1.3) is in the other wave parameters.

3. Let us now turn to Eq. (0.4), which, like (0.3), possesses a three-parameter Lie group. The Lie algebra is generated by the infinitesimal operators

$$X_1 = \frac{\partial}{\partial t},\quad X_2 = \frac{\partial}{\partial x},\quad X_3 = t\frac{\partial}{\partial t} + 3x\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u} \tag{3.1}$$

The difference as compared with (2.1) is that the dilatation operator X_3 acts on all the variables. This results in the optimum system consisting of four one-parameter subgroups:

$$X_1,\ X_2,\ X_3,\ X_4 = X_1 + \alpha X_2 \tag{3.2}$$

Subgroups X_1 and X_2 yield the trivial solution. The similarity solutions of Eqs. (0.1), (0.3), and (0.4) have different functional dependences on the variables x and t, since the dilatation subgroups have different coefficients of dilatation. The similarity solution can be sought in the form $u = t^2 U(x/t^3)$. The equation for $U(\lambda)$ is

$$\beta\left(27\lambda^3\frac{d^3U}{d\lambda^3} + 54\lambda^2\frac{d^2U}{d\lambda^2} + 6\lambda\frac{dU}{d\lambda}\right) + 3\,\frac{\lambda}{U}\frac{dU}{d\lambda} - \frac{dU}{d\lambda} - 2 = 0 \tag{3.3}$$

As in the case of (1.4), the solution of this equation is not expressible in terms of algebraic functions.

A typical feature of (0.4) is that the shape of the stationary wave is not described by an elliptic function. The wave profile can be expressed implicitly in terms of a single quadrature:

$$x = \alpha t + (\alpha\beta)^{3/2}\int\frac{du}{\sqrt{u^2 - 2\alpha u \ln u + c_1 u + c_2}} \tag{3.4}$$

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